A computer environment to encourage versatile understanding of algebraic equations

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Learning algebra is very difficult for many students, and one of the major obstacles identified in the research literature is a 'cognitive gap' presented by equations of the form ax+b=cx+d, which require operation on the variable for solution. In this small-scale, preliminary study we assessed the value of a computer environment which enables students gradually to investigate algebraic expressions and equations. The initial results are very encouraging and suggest that this may be a way to assist students to learn algebra in a versatile way.

Background

There is a wide range of research informing us that children find great difficulty in understanding of what has been called the algebra of generalised arithmetic (e.g. Küchemann, 1981; Wagner, Rachlin & Jensen, 1984; MacGregor & Stacey, 1997). One of the problems with understanding how to solve linear equations is that, prior to the introduction of algebra, students become accustomed to working in an arithmetic environment where they solve problems by producing a numerical 'answer', leading to the expectation that the same will be true for algebra. Thus, for many the '=' sign is an indication that an answer is to be expected (Kieran, 1981), leading them to implement arithmetic or informal methods in the solution of algebraic equations. However, when they are faced with an equation of the form $ax+b=cx+d$, which requires one to operate on the unknown, they often face an insurmountable obstacle. Herscovics and Linchevski (1994) have described this inability of students to operate with or on the unknown, as characterising a demarcation between arithmetic and algebra, what they term a *cognitive gap*, based on the *didactic* cut of Filloy and Rojano (1984), and they concluded that many students were reduced to performing meaningless operations on symbols they did not understand. The question is, how can one improve conceptual understanding of equation in a way which addresses obstacles like this? Linchevski and Herscovics (1996) report on their teaching experiment using decomposition and cancellation of identical terms for solving linear equations, and while they experienced a measure of success, they found "that some of these obstacles are rather robust and perhaps should not be dealt with incidentally. . . " (*ibid*, p. 39). Pirie and Martin (1997, p. 141) believe that the obstacle results from teaching algebra "through appeal to arithmetic parallel thinking" and they also describe a method resulting in some success, using a method based on grouping singleton boxes as multiples of variables.

We believe that an important part of the learning process lies in recognising that mathematical objects have two distinct but complementary faces, namely as processes or objects, or in the words of Sfard (1991), they possess operational and structural aspects. Dubinsky & Lewin (1986) relate these by discussing how learners *encapsulate* processes as objects. In a later revision of this, designated APOS theory, Cottrill *et al.* (1996) discuss how actions on objects become interiorised as processes, which in turn may be encapsulated as objects. This latter stage occurs when "the individual becomes aware of the totality of the process, realizes that transformations can act on it, and is able to construct such transformations." (*ibid*, p. 171). Concepts may then be viewed as schematic constructions comprising actions, processes and objects. Sadly, many students fail to encapsulate algebraic processes as objects and so are left with a *process-oriented* (Thomas, 1994; Kota & Thomas, 1998) view of algebra rather than progressing to the point where they can think in a *versatile* manner (Tall & Thomas, 1991; Hong & Thomas, 1998). This *versatile* perspective is one where, through encapsulation, learners attain a global view of the objects and/or processes comprising a concept, alongside its components, or constituent process(es), which they can think sequentially about, and inter-relate these two perspectives. In contrast, thinking primarily in terms of mathematical processes structures thinking towards procedural methods, algorithms and presentations (Thomas, 1994).

Attaining a versatile, conceptual understanding of algebra involves learning its concepts, which are primarily (at least initially) variable, expression, function and equation, via actions which enable one to construct a view of them as process or object. Applying this to equations, versatile learning requires students to be able to investigate processes of building and solving equations, but at the same time to build a global view of equation which sees it as an object based on the equivalence of two functions or expressions. This property of equivalence is necessary to underpin the processes that students engage in, such as adding the same terms to both sides and/or cancelling equivalent terms. Hence, equations may be categorised as *procepts* (Gray & Tall, 1991, 1994), a combination of mathematical symbols, a process and an object, and student thinking should be directed toward activities which build this perspective.

This research sought to investigate whether it is possible to improve students' proceptual understanding of equations by giving them an environment in which they could manipulate examples, predict and test results, and gain experiences on which a versatile understanding of expression and equation as process and object could be built.

Method

This was very much a preliminary, small scale study involving case studies of a small number of students. While pre- and post-tests were used, the main aim was to investigate individual student's improvements in understanding. The second named author was the class teacher of all the students and conducted the tests and interviews.

Subjects

Twelve students of similar ability on standardised testing (TOSCA Stanines of 7 to 9+) were chosen; six from year 7 (age 11 years) and six from year 8 (age 12 years), with three females and three males in each group. This selection took place in May 1997, allowing for the year 7 children to be well settled into their school year. The year 7 students had not been taught any algebra at all, while the year 8 students had been introduced to algebra the previous year. In the event, of the initial twelve students selected to participate, one of the year 7 group was unavailable to participate in the pre-test and so was removed from the study group, reducing it to eleven students.

Procedure

Initially, all the students were briefly interviewed individually about their understanding of algebra, and after their interview they were given a pre-test of 40 linear equations to solve, based on questions from a test used by Herscovics and Linchevski (1994). In this test the linear equations were grouped into 8 sections, increasing in difficulty, beginning with $14 + n = 43$ and ending with $5n + 12 = 3n + 24$, an equation spanning the cognitive gap. The comparison with Herscovics and Linchevski (1994) was useful because their research highlighted both the relative difficulty of the questions and the types of solution procedures used by students. All the students were allowed to use a calculator when completing the tests.

Each student was supplied with their own disk copy of a computer software program, entitled *Dynamic Algebra,* comprising an environment for investigating algebraic concepts, covering the concepts of expression and equation in algebra, and this was used as a teaching tool. They were initially shown how to use the program and were then instructed to trial it on their own, with any queries being followed up on an individual basis. The students then spent the next six weeks using the program and working through its various levels. Since they all had access to a computer at home, they were also encouraged to use the program for at least fifteen minutes per night. The children also participated in their daily classroom mathematics programme, and at the end of the six weeks they were given a post-test comprising the same test used for the pre-test.

The Dynamic Algebra computer environment

The *Dynamic Algebra* program comprises three basic areas: substitution of a variable into an expression; equivalence of expressions; and equations. Each of these areas is further broken down into three categories: machine choice; abstract choice and a test situation to check understanding of the particular area. The program is built on the concept of variable as a named store, one which had worked well in other contexts (Tall & Thomas, 1991; Graham & Thomas, 1997), and students see a variable label under a box containing its current value. Figure 1 shows a sample screen from the trial and error section on equation solving. This section of the program is extremely important since it combines the above model of variable with the idea of substitution in an expression (or function value) and equivalence of both sides of an equation. Students investigate solutions by entering values for the variable, here u, until both sides are seen to be equal in value, that is the difference between the two sides is zero.

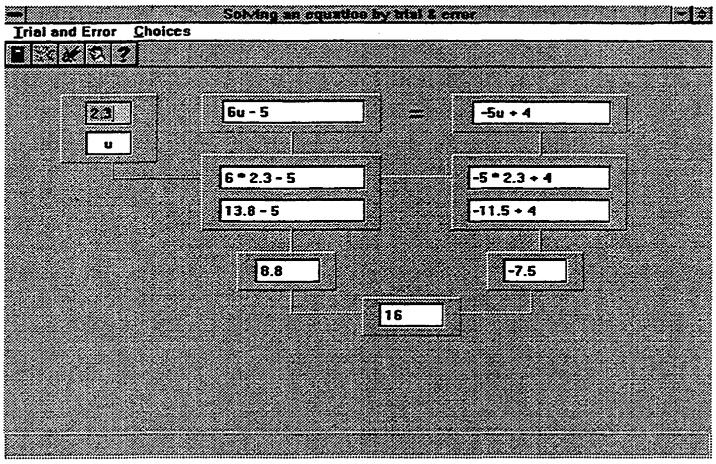

Figure 1: A screen showing a trial and error approach to solving an equation

All the boxes in this mode other than the variable's value are completed by the program, which shows both the full expressions with substitution of the variable's value, and the simplified versions. Although the equations are generated automatically, the difficulty level of the linear function used may be chosen, and there is the option of user entry of expressions and equations.

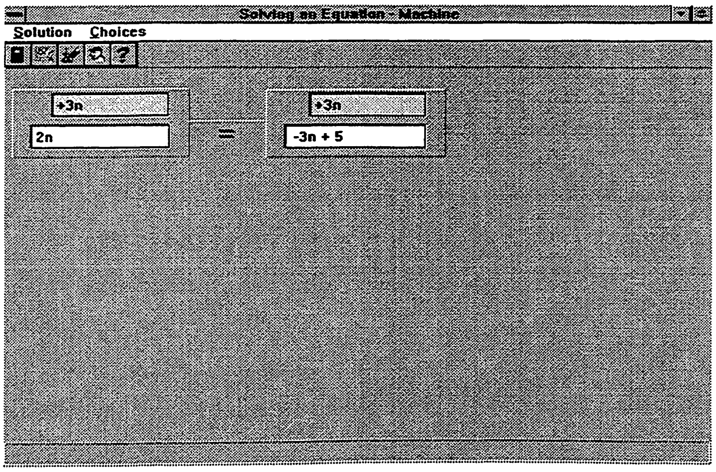

Figure 2: A solution to an equation involving the same operation on both sides

Figure 2 shows a sample from the section on solving equations by adding like terms to both sides. At any point the program constrains the user to entry of a 'correct' multiple of the variable or a constant. Should a student be unable to continue at any particular point in a solution then they may click on the *coach* button and get the assistance of an imaginary coach who fills in the next box. A 'smiley face' greets a correct solution (see figure 3).

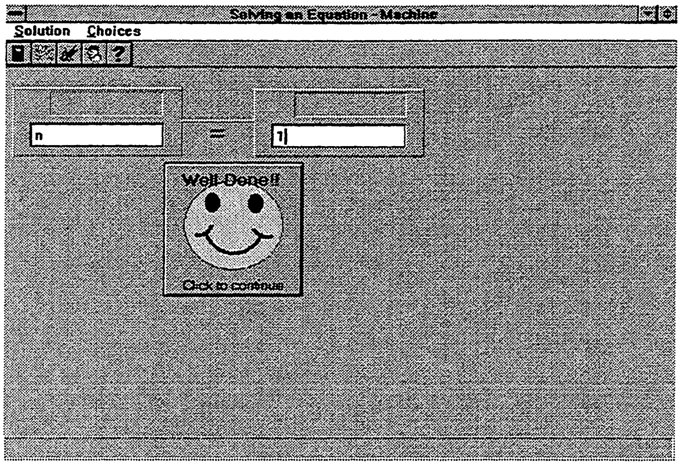

Figure 3: A screen showing a successful solution to an equation

In this way the students were able to work through many of the areas of the computer environment on their own.

Results

Year 7 students

In their preliminary interview the students were asked what they knew about $3n$ to gauge their knowledge of algebra. The year 7 students, who were understood not to have studied any algebra, were naturally hesitant in their responses. One student thought that it could be 3 x 3, but most had no idea, confirming that they had no prior knowledge of algebra. To confirm their responses the students were asked to replace the n with a 2 or a 5, and the responses were similar to the previous question, with the students responding 32 or 35, substituting the number without considering that any multiplication was required. One might expect this lack of understanding to be a severe handicap in solving equations, but on the pre-test, these students, coded A to E, correctly solved 22, 29, 10, 31 and 24 of the 40 questions respectively. Three of the students B, D and E, were even able to solve some equations such as:

$$n + n = 76, \quad 4n + 17 = 65 \quad \text{and} \quad 5n + n = 78$$

even though they had been unable to attach the correct meaning to a multiple of n. For the first of these questions, this phenomenon was noticed by Linchevski and Hescovics (1996, p. 43) who commented that for $n + n = 76$ etc.

> Most students immediately divided 76 or 50 by 2. These results indicate that when the terms in the unknown are singletons, i.e. without any coefficient, the majority of students have a natural tendency to mentally group the terms in the solution process.

However, it is very surprising that this could be extended to the second and third questions, which appear to require understanding what the $5n$ means. It would seem that these students used a combination of trial and error substitution, invented and informal procedures, possibly adding the coefficients and dividing by the total in some questions, correctly emulating algebraic procedures. Certainly, student B knew what $4n$ meant by the time of the pre-test, and was also using trial and error substitution, since, for the question $3n + 5n + 4n = 19$, he tried $n = 1.5$ and wrote $4.5 + 7.5 + 6 = 18$, then tried 1.6 and got 19.2 and then 1.55 getting 18.6, so as his answer he wrote, 'More than 1.55, less than 2', since he was concerned that if he wrote 'between 1.55 and 1.6' this would not be considered as an answer. He clearly did not use the calculator to find 19/12. Students A and C, as one would expect, were unable to answer correctly any question involving a multiple of n, or more than a single variable in a question. For the final three questions, each of which had an unknown on both sides of the equation, there was only one correct answer, from student B, and it turned out, on investigation, that this was a fortuitous guess.

After their work on the computer, as table 1 shows, their marks had improved significantly to 40, 38, 39, 40 and 35 respectively.

Table 1: An analysis of the pre- and post-test mean scores of the year 7 and year 8 students

Year	Pre-Test	Post-Test	t	p
Year 7	23.2	38.4	3.97	<0.05
Year 8	31.5	38.8	3.71	<0.05
Combined	27.7	38.6	4.76	<0.001

Student A, who improved from 22 to 40, and who could not deal with any equation other than those with a single n, not only got all the questions correct but had built an understanding of the need to treat each side of an equation the same. For the equation $n + 15 = 4n$, he was able to write $3n = 15$, $n = 5$, immediately, but for the final question: $5n + 12 = 3n + 24$, he wrote:

$$2n + 12 - 12 = 24 - 12; \qquad 2n = 12; \qquad n = 6$$

and when asked afterward what he had done he dictated the 'missing' first line,

$$5n - 3n + 12 = 3n - 3n + 24.$$

This remarkable improvement, to a point where he had avoided the cognitive gap, was emulated by student C, who had the lowest pre-test score of 10, and who improved to 39. For the equation, $n + 15 = 4n$, she wrote:

$-n + n + 15 = 4n - n;\quad 15 = 3n;$ and the answer as 5,

with no concern about the $-n$ appearing first. Unfortunately she was somewhat too rushed to finish the final question, but she did enough to show that she too had constructed the principle of treating both sides the same, writing,

$$5n - 3n + 12 = 3n - 3n + 24$$
$$2n + 12 = 24$$
$$2n + 12 - 12 =$$
$$2n = 44; \qquad n = 22$$

The blank in the third from last line seems to be an indication that she had made a mistake rather than having an error in her understanding.

Two limitations of the test instrument which emerged from this study were: the questions were not difficult enough at the post-test, in particular there should have been more questions with variables on both sides of the equation; and students were not required to show their working. A consequence of these two limitations was that most wrote down only the answers, or if they did show working they erased it afterward. This last point was true of all three students, B, D and E. However, student B improved on the question $3n + 5n + 4n = 19$ so that he was able to write 1.58 as the solution, probably by adding the ns and dividing 19 by 12. He also wrote some indication that he may have been performing the same operation on both sides when he wrote, $5n - 3n + 12$ as partial working on the final question, $5n + 12 = 3n + 24$ (note that they had not been exposed to 'change side, change sign' in any form). He commented that "The program was easy to use and fun." Student D got all the questions correct, but with no indication of her methods , and the same was true of student E, however she thought that it was "a good idea to have a programme on the computer. It's good because you can choose the level which is right for you. Overall I thought it was really good."

Certainly, from a base of no knowledge at all of algebra we can say that the level of understanding and ability to solve the equations at the end of the six week period on the computer was very pleasing.

Year 8 students

Since the year 8 students, coded F to K, had already been exposed to an introduction to algebra, when asked in their preliminary interview what they knew about $3n$, four of the six students replied that it was the same as saying 3 x n. One student, H, was not sure and another suggested that it could be 3 x 3. On substituting 2 and 5, most replied that it was 3 x 2 = 6, or 3 x 5 = 15, although H did respond that it was 32 or 35.

These students, who scored 38, 29, 27, 26, 37 and 32 respectively on their pre-test appeared to be too good to gain much from their use of the computer environment. However, as table 1 shows, their mean score did improve significantly, and we also noted that the methods they used improved considerably. Student F used trial and error and checking by substitution on the pre-test and although she still used this method on some questions on the post-test, when she scored 39, for the equation: $3n + 5n + 4n = 19$, she wrote:

$$12n = 19, \qquad n = 19/12,$$

this being a question she did not attempt on the pre-test. She also described the program as "quite helpful". Student I, also wrote that the program "really improved my understanding of algebra. . . They should keep this program". He got 39 correct, making only a calculator entry mistake on an early question. Student J, who managed 37 correct on the pre-test, was, even then able to solve $n + 15 = 4n$, by writing:

$$15 = 3n; \qquad n = 5,$$

although for the final question he asked: "Is this saying that $5n + 12$ is the same as $3n + 24$?", and was clearly bemused by it and unable to make any attempt. On the post-test he scored 39, again making only a calculator entry mistake on an early question but this time correctly solving $5n + 12 = 3n + 24$, although with no indication of method. However, his comments on the program included the encouraging remark: "Helped you learn formulas eg What you do to one side you do to the otherside [sic]."

The remaining three students, G, H and K showed the best improvements in understanding. G used guess and substitution on all her questions on the pre-test, getting 29 correct, but commented on questions with more than one n, when interviewed afterward, "Does n have a variety of numbers or are they the same?"

After the computer work she appeared to have no problem with this and for the final question, $5n + 12 = 3n + 24$ wrote:

$$5n - 3n + 12 = \cancel{3n} - \cancel{3n} + 24; \quad 2n + 12 = 24; \quad 2n + \cancel{12} - \cancel{12} = 24 - 12;$$

$$2n = 12; \qquad n = 6.$$

This method and the 'cancelling' of the $3n$ and the 12 are significant steps forward. G commented on her computer work: "I quite liked simplification. Algebra is easier on the computer than on paper. . . I worked by myself how some of these equations as Mrs Hall did not show me." H, with 27 on the pre-test, used guess and substitute, and could not solve any equation with a multiple of n in it. On the post-test she got 38 correct, including all the final three questions, for one of which she wrote:

$$4n + 9 = 7n; \qquad \cancel{4n} - \cancel{4n} + 9 = 7n - 4n; \qquad 9 = 3n; \qquad n = 3,$$

cancelling the $4n$s at stage one. Of the algebra program she said "If I didn't know something then the answer guy was a big help. . . The way it took you through step by step was good. Overall I think the algebra program was good." Finally, student K, on the pre-test scored 32 and was able to add like terms and use inverse operations in his solutions, for example, on $3n + 4n = 35$ he used 35/7. He was not able to answer the final question. After the computer work he got everything correct and worked through the final question exactly the same as G above, but without the physical cancelling of terms. Again, these gains are worthy of note, and lead us to believe that the program has value in helping students build understanding in algebra.

Not everything about the computer environment was to the liking of the students. They wanted to see some improvements and these included: the need for an in-built calculator; "you should be able to get the answers wrong in the tests"; not enough instructions; the 'smiley face' was annoying.

Conclusion

As we have made clear, this was a preliminary study on the use of the *Dynamic Algebra* environment. It was pleasing to note that, not only did the students as groups perform significantly better in terms of the overall number of questions they got correct, but they had developed a more versatile view of equation. This was shown by the clear evidence that after the computer work 6 of the 11 students were known to be using a method of solution where they applied the same operation to both sides of the equation and 'cancelled' terms, even operating on the variable and 8 of the 11 were able to solve the equation $5n + 12 = 3n + 24$. It appears that they had successfully avoided the cognitive gap described by Herscovics and Linchevski (1994). The students found the programme interesting and stimulating, but also pointed out improvements, including the need for written instructions (the program has a built-in Help menu). These results have been very encouraging and we are now designing a larger study to confirm them, and to take into account the lessons learned.

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